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A new Kontorowich–Lebedev-like transformation

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Abstract. In this paper a new integral with respect to the index of Bessel functions of the first kind is evaluated.

1. Introduction

The equation of a massive scalar field in a two-dimensional Milne's universe [1, 2] is

$$\left(\eta^2\frac{\partial^2}{\partial\eta^2}+\eta\frac{\partial}{\partial\eta}-\frac{\partial^2}{\partial\xi^2}+\eta^2m^2\right)\Phi(\eta,\xi)=0. \tag{1.1}$$

In order to quantize this field, a complete set of mode solutions of (1.1) is required. Sommerfeld [3] proposed

$$v_\lambda(\eta,\xi)=\frac{-\mathrm{i}}{2}(\sinh(\pi|\lambda|))^{-1/2}\,\mathrm{e}^{\mathrm{i}\lambda\xi}J_{-\mathrm{i}|\lambda|}(m\eta) \tag{1.2a}$$

$$v^*_\lambda(\eta,\xi)=\frac{+\mathrm{i}}{2}(\sinh(\pi|\lambda|))^{-1/2}\,\mathrm{e}^{-\mathrm{i}\lambda\xi}J_{+\mathrm{i}|\lambda|}(m\eta) \tag{1.2b}$$

as a complete set of mode solutions, with $\lambda\in\mathbb{R}$ and J_ν Bessel functions of the first kind with order ν[4, 5]. Without loss of generality we will take the mass of the quanta of the field to be $m=1$.

In the calculation of the propagators of the field which satisfies (1.1), quantized by the mode-solutions (1.2), a new integral was found. It is a Kontorowich–Lebedev-like transformation:

$$\begin{aligned}I(x,y_1,y_2)&=P.V.\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\sinh(\pi\lambda)}J_{-\mathrm{i}\lambda}(y_1)J_{+\mathrm{i}\lambda}(y_2)\\&\quad-P.V.\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\sinh(\pi\lambda)}J_{+\mathrm{i}\lambda}(y_1)J_{-\mathrm{i}\lambda}(y_2)\\&=\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\sinh(\pi\lambda)}[J_{-\mathrm{i}\lambda}(y_1)J_{+\mathrm{i}\lambda}(y_2)-J_{+\mathrm{i}\lambda}(y_1)J_{-\mathrm{i}\lambda}(y_2)].\end{aligned} \tag{1.3}$$

Defining

$$f(\lambda,a,b,c)=\frac{\mathrm{e}^{\mathrm{i}\lambda a}}{\sinh(\pi\lambda)}J_{-\mathrm{i}\lambda}(b)J_{+\mathrm{i}\lambda}(c) \tag{1.4}$$

the integral to be evaluated becomes

$$I(x, y_1, y_2) = P.V. \int_{-\infty}^{+\infty} d\lambda\, f(\lambda, x, y_1, y_2) - P.V. \int_{-\infty}^{+\infty} d\lambda\, f(\lambda, x, y_2, y_1). \tag{1.5}$$

2. The evaluation of $P.V. \int_{-\infty}^{+\infty} d\lambda\, f(\lambda, a, b, c)$

The function $f(\lambda, a, b, c)$ is analytic with respect to λ in $\mathbb{C}/\{ik: k \in \mathbb{Z}\}$. In the points $\lambda = ik$ $(k \in \mathbb{Z})$, either $f(\lambda \ldots)$ has a first-order pole with

$$\operatorname{Res}(f(\lambda, a, b, c), \lambda = ik) = \frac{e^{-ka}}{\pi} J_k(b) J_k(c) \tag{2.1}$$

or $J_k(a)J_k(b) = 0$ and f can be analytically continued to $\lambda = ik$.

Two distinct contours, $C(n, \varepsilon)$ and $C'(n, \varepsilon)$ will be used to perform the integration (see figures 1 and 2). C_4 and C_4' are semi-circles centred at $\lambda = 0$ and with radius $n + \frac{1}{2}$, $n \in \mathbb{N}$. C_2 and C_2' are semi-circles centred at $\lambda = 0$ and with radius ε, $0 < \varepsilon < 1$.

It is convenient to employ the notation:

$$(z)_0 = 1 \qquad (z)_1 = z \qquad (z)_k = z(z+1)\ldots(z+k-1). \tag{2.2}$$

The expansion of J in a power series yeilds:

$$J_\nu(y) = (\tfrac{1}{2}y)^\nu \sum_{k=0}^{k=\infty} \frac{(-y^2/4)^k}{k!\Gamma(\nu+k+1)} = \frac{(\frac{1}{2}y)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{k=\infty} \frac{(-y^2/4)^k}{k!(\nu+1)_k} \tag{2.3}$$

so that $f(\lambda, a, b, c)$ can be expressed as

$$f(\lambda, a, b, c) = \frac{e^{i\lambda a}}{\sinh(\pi\lambda)} \frac{(c/b)^{i\lambda}}{\Gamma(i\lambda+1)\Gamma(-i\lambda+1)} \sum_{k=0}^{k=\infty} \frac{(-c^2/4)^k}{k!(i\lambda+1)_k} \times \sum_{k=0}^{k=\infty} \frac{(-b^2/4)^k}{k!(-i\lambda+1)_k}. \tag{2.4}$$

Using the relationship $\Gamma(-i\lambda+1)\Gamma(i\lambda+1) = \pi\lambda/\sinh(\pi\lambda)$,

$$f(\lambda, a, b, c) = \frac{(e^a c/b)^{i\lambda}}{\pi\lambda} \sum_{k=0}^{k=\infty} \frac{(-c^2/4)^k}{k!(i\lambda+1)_k} \sum_{k=0}^{k=\infty} \frac{(-b^2/4)^k}{k!(-i\lambda+1)_k}. \tag{2.5}$$

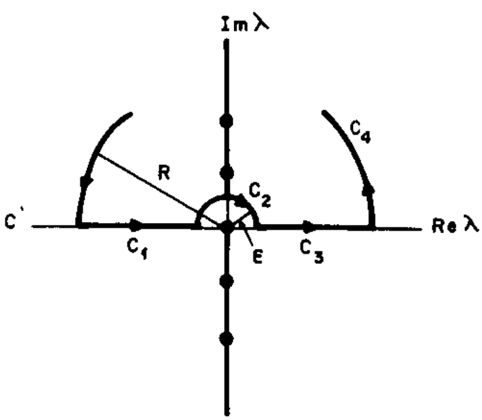

Figure 1

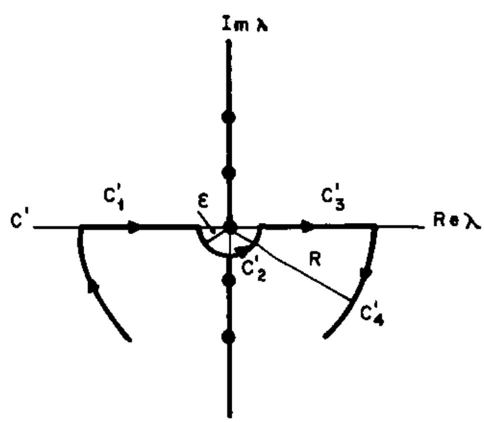

Figure 2

Using the relationship ($\lambda \in \mathbb{R}$):

$$\left|\sum_{k=1}^{k=\infty} \frac{u^k}{k!(-i\lambda+1)_k}\right| \leqslant \frac{e^{|u|}}{|i\lambda+1|}$$

we can express $f(\lambda, a, b, c)$ as

$$f(\lambda, a, b, c) = \frac{(e^a\, c/b)^{i\lambda}}{\pi\lambda} + g(\lambda, a, b, c)$$

with $|g(\lambda, a, b, c)| \leqslant M/(\lambda^2)$ for some $M \in \mathbb{R}$.

Now it is clear that, whenever $a, b, c \in \mathbb{R}$, $b, c > 0$ and $e^a\, b/c$ is not equal 1, the integral $\int_\varepsilon^\infty f(\lambda, a, b, c)\, d\lambda$ does exist ($\varepsilon > 0$).

If $\lambda \in C_4$ or $\lambda \in C_4'$ then $|(\pm i\lambda + 1)_k| \geqslant 1/2^k$ and

$$|f(\lambda, a, b, c)| \leqslant \frac{(e^a\, c/b)^{-\text{Im}(\lambda)}}{\pi|\lambda|}\, e^{(c^2+b^2)/2} \qquad |\lambda| = n + \tfrac{1}{2},\ n \in \mathbb{N}. \tag{2.6}$$

Employing the parameterizations:

$$\lambda = (n + \tfrac{1}{2})\, e^{i\theta} \qquad 0 \leqslant \theta \leqslant \pi \text{ in } C_4 \tag{2.7a}$$

$$-\pi \leqslant \theta \leqslant 0 \text{ in } C_4' \tag{2.7b}$$

the absolute value of the integral of f over C_4 and C_4' can be estimated:

$$\left|\int_{C_4} f(\lambda, a, b, c)\, d\lambda\right| \leqslant \frac{e^{(c^2+b^2)/2}}{\pi}\, 2 \int_0^{\pi/2} (e^a\, c/b)^{-(n+\frac{1}{2})\sin(\lambda)}\, d\lambda \tag{2.8a}$$

$$\left|\int_{C_4'} f(\lambda, a, b, c)\, d\lambda\right| \leqslant \frac{e^{(c^2+b^2)/2}}{\pi}\, 2 \int_0^{\pi/2} (e^a\, c/b)^{+(n+\frac{1}{2})\sin(\lambda)}\, d\lambda \tag{2.8b}$$

and

$$\lim_{n\to\infty} \int_{C_4} f(\lambda, a, b, c)\, d\lambda = 0 \qquad \text{if } e^a\, c/b > 1 \tag{2.9a}$$

$$\lim_{n\to\infty} \int_{C_4'} f(\lambda, a, b, c)\, d\lambda = 0 \qquad \text{if } e^a\, c/b < 1. \tag{2.9b}$$

Using the residue theorem we get

$$\int_{C(n,\varepsilon)} f(\lambda, a, b, c)\, d\lambda = 2\pi i \sum_{k=1}^{k=n} \frac{e^{-ka}}{\pi} J_k(b) J_k(c) \tag{2.10}$$

$$\int_{C'(n,\varepsilon)} f(\lambda, a, b, c)\, d\lambda = -2\pi i \sum_{k=-1}^{k=-n} \frac{e^{-ka}}{\pi} J_k(b) J_k(c). \tag{2.11}$$

If $e^a\, c/b > 1$,

$$\lim_{\substack{n\to\infty_+(n\in\mathbb{N})\\ \varepsilon\to 0}} \int_{C(n,\varepsilon)} f(\lambda, a, b, c)\, d\lambda$$

$$= P.V. \int_{-\infty}^{+\infty} f(\lambda, a, b, c)\, d\lambda - \pi i\, \text{Res}(f, 0)$$

$$= 2\pi i \sum_{k=1}^{k=\infty} \frac{e^{-ka}}{\pi} J_k(b) J_k(c). \tag{2.12}$$

Therefore

$$P.V. \int_{-\infty}^{+\infty} f(\lambda, a, b, c)\, d\lambda = 2i\left[\tfrac{1}{2}J_0(b)J_0(c) + \sum_{k=1}^{k=+\infty} e^{-ka} J_k(b)J_k(c)\right] \qquad (2.13a)$$

if $e^a c/b > 1$. The same reasoning, using $C'(n, \varepsilon)$ when $e^a c/b < 1$, gives

$$P.V. \int_{-\infty}^{+\infty} f(\lambda, a, b, c)\, d\lambda = -2i\left[\tfrac{1}{2}J_0(b)J_0(c) + \sum_{k=-1}^{k=-\infty} e^{-ka} J_k(b)J_k(c)\right] \qquad (2.13b)$$

if $e^a c/b < 1$.

3. The evaluation of $I(x, y_1, y_2)$

From (2.13*a*) and (2.13*b*), it follows that $I(x, y_1, y_2) = 0$ if $e^{-x} < y_1/y_2 < e^x$ and if $e^x < y_2/y_1 < e^{-x}$, while

$$I(x, y_1, y_2) = 2i \sum_{k=-\infty}^{k=+\infty} e^{-kx} J_k(y_1)J_k(y_2) \qquad \text{if } y_1/y_2 < e^{-x} \text{ and } y_1/y_2 < e^x$$

$$I(x, y_1, y_2) = 2i \sum_{k=-\infty}^{k=+\infty} e^{-kx} J_k(y_1)J_k(y_2) \qquad \text{if } y_1/y_2 > e^{-x} \text{ and } y_1/y_2 > e^x.$$

It is known [6] that

$$J_0((b^2 + c^2 - 2bc\cosh(a))^{1/2}) = \sum_{k=-\infty}^{k=+\infty} e^{-ka} J_k(b)J_k(c). \qquad (3.1)$$

Defining

$$\sigma(x, y_1, y_2) = y_1^2 + y_2^2 - 2y_1y_2\cosh(x) \qquad (3.2)$$

algebraic manipulations of the preceding formula give

$$\sigma(x, y_1, y_2) = -y_1y_2 e^{-x}(e^x y_2/y_1 - 1)(e^x y_1/y_2 - 1). \qquad (3.3)$$

So if $\sigma < 0$ then $e^{-x} < y_1/y_2 < e^x$ or $e^x < y_2/y_1 < e^{-x}$ and $I(x, y_1, y_2) = 0$.

$$I(x, y_1, y_2) = 0 \qquad \sigma < 0. \qquad (3.4)$$

If $\sigma(x, y_1, y_2) > 0$ and $y_2 > y_1$ then $y_1/y_2 < e^{-x}$ and $y_1/y_2 < e^x$, and

$$I(x, y_1, y_2) = 2iJ_0(\sigma(x, y_1, y_2)^{1/2}) \qquad \sigma > 0, y_2 > y_1. \qquad (3.5)$$

If $\sigma(x, y_1, y_2) > 0$ and $y_1 > y_2$ then $y_1/y_2 > e^{-x}$ and $y_1/y_2 > e^x$, and

$$I(x, y_1, y_2) = -2iJ_0(\sigma(x, y_1, y_2)^{1/2}) \qquad \sigma > 0, y_2 < y_1. \qquad (3.6)$$

Defining

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases} \qquad (3.7a)$$

$$S(t) = \begin{cases} -1 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases} \qquad (3.7b)$$

we have

$$I(x, y_1, y_2) = 2iH(\sigma)S(y_2 - y_1)J_0(\sigma^{1/2}) \qquad \sigma = y_1^2 + y_2^2 - 2y_1y_2\cosh(x). \qquad (3.8)$$

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